Theory of Equations Part-1

1. Every n[th] degree equation has exactly n roots real or imaginary.

2. Relation between, roots and coefficients of an equation.

(i) If α, β, γ are the roots of $x^3 + p_1x^2 + p_2x + p_3 = 0$ the sum of the roots

$s_1 = \alpha + \beta + \gamma = -p_1.$

Sum of the products of two roots taken at a time $s_2 = \alpha\beta + \beta\gamma + \gamma\alpha = -p_2.$

Product of all the roots, $s_3 = \alpha\beta\gamma = -p_3.$

(ii) If $\alpha, \beta, \gamma, \delta$ are the roots of $x^4 + p_1x^3 + p_2x^2 + p_3x + p_4 = 0$ then

Sum of the roots $s_1 = \alpha + \beta + \gamma + \delta = -p_1.$

$s_2 = \alpha\beta + \alpha\gamma + \alpha\delta + \beta + \beta\delta + \gamma\delta = p_2.$

Sum of the products of roots taken three at a time

$s_3 = \alpha\beta\gamma + \beta\gamma\delta + \gamma\delta\alpha + \delta\alpha\beta = -p_3.$

Product of the roots, $s_4 = \alpha\beta\gamma\delta = p_4.$

3. For the equation $x^n + p_1 x^{n-1} + p_2 x^{n-2} + \ldots + p_n = 0$

i) $\Sigma\alpha^2 = p_1^2 - 2p_2$

ii) $\Sigma\alpha^3 = -p_1^3 + 3p_1p_2 - 3p_3$

iii) $\Sigma\alpha^4 = p_1^4 - 4p_1^2p_2 + 2p_2^2 + 4p_1p_3 - 4p_4$

iv) $\Sigma\alpha^2\beta = 3p_3 - p_1p_2$

v) $\Sigma\alpha^2\beta\gamma = p_1p_3 - 4p_4$

Note: For the equation $x^3 + p_1x^2 + p_2x + p_3 = 0$ $\Sigma\alpha^2\beta^2 = p_2^2 - 2p_1p_3$

4. To remove the second term from a n^{th} degree equation, the roots must be diminished by $\frac{-a_1}{na_0}$ and the resultant equation will not contain the term with x^{n-1}.

5. If α_1 , α_2 , ...α_n are the roots of f(x) = 0, the equation

i) Whose roots are $\frac{1}{\alpha_1}, \frac{1}{\alpha_2} .... \frac{1}{\alpha_n}$ is $f\left(\frac{1}{x}\right) = 0$.

ii) Whose roots are $k\alpha_1$, $k\alpha_2$,...,$k\alpha_n$ is $f\left(\frac{x}{k}\right) = 0$.

iii) Whose roots are $\alpha_1 - h, \alpha_2 - h, ...\alpha_n - h$ is f(x+h) = 0.

iv) Whose roots are $\alpha_1 + h$, $\alpha_2 + h$, ...$\alpha_n + h$ is f(x-h) = 0.

v) Whose roots are α_1^2, α_2^2.... α_n^2 is $f\left(\sqrt{y}\right) = 0$

6. In any equation with rational coefficients, irrational roots occur in conjugate pairs.

7 . In any equation with real coefficients, complex roots occur in conjugate pairs.

8. If α is r $^-$ multiple root of f(x) = 0, then α is a (r-1) - multiple root of $f^1(x)=0$ and (r-2) - Multiple root of $f^{11}(x) = 0$ and non multiple root of $f^{r-1}(x) = 0$.

9. i) If $f(x) = x^n + p_1x^{n-1} + ... + p_{n-1}x + p_n$ and f(a) and f(b) are of opposite sign, then at least one real root of f(x) =0 lies between a and b.

10. (a) For a cubic equation, when the roots are

(i) In A.P., then they are taken as $a-d, a, a+d$

(ii) In G.P., then are taken as $\frac{a}{r}, a, ar$

(iii) In H.P., then they are taken as $\frac{1}{a-d}, \frac{1}{a}, \frac{1}{a+d}$

11. (b) For a bi quadratic equation, if the roots are

(i) In A.P., then they are taken as $a-3d, a+d, a+3d$.

(ii) In G.P., then they are taken as $\frac{a}{d^3}, \frac{a}{d}, ad, ad^3$

(iii) In H.P., then they are taken as $\frac{1}{a-3d}, \frac{1}{a-d}, \frac{1}{a+d}, \frac{1}{a+3d}$.

12. (i) If an equation is unaltered by changing x into $\frac{1}{x}$, then it is a reciprocal equation.

(ii) A reciprocal equation $f(x) = p_0x^n + p_1x^{n-1} + ... + p_n = 0$ is said to be a reciprocal equation of first class $p_i = p_{n-i}$ for all i.

(iii) A reciprocal equation $f(x) = p_0x^n + p_1x^{n-1} + ... + p_n = 0$ is said to be a reciprocal equation of second class $p_i = p_{n-i}$ for all i.

(iv) For an odd degree reciprocal equation of class one, -1 is a root and for an odd degree reciprocal equation of class two, 1 is a root.

(v) For an even degree reciprocal equation of class two, 1 and -1 are roots.

13. If $f(x)=0$ is an equation of degree 'n' then to eliminate r[th] term, $f(x)=0$ can be transformed to $f(x+h)=0$ where h is a constant such that $f^{(n-r+1)}(h)=0$ i.e., $(n-r+1)^{th}$ derivative of $f(h)$ is zero.

Very Short Answer Questions

1. Form polynomial equations of the lowest degree, with roots as given below.

(i) $1, -1, 3$

Equations having roots α, β, γ is $(x-\alpha)(x-\beta)(x-\gamma)=0$

Sol: Required equation is

$$(x-1)(x+1)(x-3)=0$$

$$\Rightarrow (x^2-1)(x-3)=0$$

$$\Rightarrow x^3-3x^2-x+3=0$$

(ii) $1 \pm 2i, 4, 2$

In an equation imaginary roots occur in conjugate pairs.

Sol: Equation having roots $\alpha, \beta, \gamma, \delta$ is $(x-\alpha)(x-\beta)(x-\gamma)(x-\delta)=0$

Required equation is

$$[x-(1+2i)][x-(1-2i)](x-4)(x-2)=0$$

$$[x-(1+2i)][x-(1-2i)]$$

$$=[(x-1)-2i][(x-1)+2i]$$

$$=(x-1)^2-4i^2$$

$$=(x-1)^2+4$$

$$=x^2-2x+1+4$$

$= x^2 - 2x + 5$

$(x-4)(x-2) = x^2 - 4x - 2x + 8$

$= x^2 - 6x + 8$

Required equation is

$(x^2 - 2x + 5)(x^2 - 6x + 8) = 0$

$\Rightarrow x^4 - 2x^3 + 5x^2 - 6x^3 + 12x^2 - 30x + 8x^2 - 16x + 40 = 0$

$\Rightarrow x^4 - 8x^3 + 25x^2 - 46x + 40 = 0$

(iii) $2 \pm \sqrt{3}, 1 \pm 2i$

Sol: Required Equation is

$\left[x-\left(2+\sqrt{3}\right)\right]\left[x-\left(2-\sqrt{3}\right)\right]$

$\left[x-(1+2i)\right]\left[x-(1-2i)\right] = 0$ ....(1)

$\left[x-\left(2+\sqrt{3}\right)\right]\left[x-\left(2-\sqrt{3}\right)\right]$

$= \left[(x-2)-\sqrt{3}\right]\left[(x-2)+\sqrt{3}\right]$

$= (x-2)^2 - 3$

$= x^2 - 4x + 4 - 3$

$= x^2 - 4x + 1$

$\left[x-(1+2i)\right]\left[x-(1-2i)\right]$

$= \left[(x-1)-2i\right]\left[(x-1)+2i\right]$

$= (x-1)^2 - 4i^2$

$$=\left(x^2-2x+1+4\right)$$

$$=x^2-2x+5$$

Substituting in (1), required equation is

$$\left(x^2-4x+1\right)\left(x^2-2x+5\right)=0$$

$$\Rightarrow x^4-4x^3+x^2-2x^3+8x^2-2x+5x^2-20x+5=0$$

$$\Rightarrow x^4-6x^3+14x^2-22x+5=0$$

(iv) $0,0,2,2,-2,-2$

Sol: Required Equation is

$$(x-0)(x-0)(x-2)(x-2)=0$$

$$\Rightarrow x^2(x+2)(x+2)=0$$

$$\Rightarrow x^2(x-2)^2(x+2)^2=0$$

$$\Rightarrow x^2\left(x^2-4\right)^2=0$$

$$\Rightarrow x^2\left(x^4-8x^2+16\right)=0$$

$$\Rightarrow x^6-8x^4+16x^2=0$$

(v) $1\pm\sqrt{3},2,5$

Sol: Required equation is

$$\left[x-\left(1+\sqrt{3}\right)\right]\left[x-\left(1-\sqrt{3}\right)\right]\left[(x-2)(x-5)\right]=0 \qquad ----(1)$$

$$\left[x-\left(1+\sqrt{3}\right)\right]\left[x-\left(1-\sqrt{3}\right)\right]=\left[(x-1)-\sqrt{3}\right]\left[(x-1)+\sqrt{3}\right]$$

$$=(x-1)^2-3=x^2-2x+1-3$$

$= x^2 - 2x - 2$

$(x-2)(x-5) = x^2 - 2x - 5x + 10$

$= x^2 - 7x + 10$

Substituting in (i) required equation is

$(x^2 - 2x - 2)(x^2 - 7x + 10) = 0$

$\Rightarrow x^4 - 2x^3 - 7x^3 + 14x^2 + 14x + 10x^2 - 20x - 20 = 0$

$\Rightarrow x^4 - 9x^3 + 22x^2 - 6x - 20 = 0$

(vi) $0, 1, -\frac{3}{2}, -\frac{5}{2}$

Sol: Required Equation is

$x(x-1)\left(x+\frac{3}{2}\right)\left(x+\frac{5}{2}\right) = 0$ ....(1)

$x(x-1) = x^2 - x$

$\left(x+\frac{3}{2}\right)\left(x+\frac{5}{2}\right) = x^2 + \frac{3}{2}x + \frac{5}{2}x + \frac{15}{4}$

$= x^2 + 4x + \frac{15}{4}$

Substituting in (i), required equation is

$\Rightarrow (x^2 - x)\left(x^2 + 4x + \frac{15}{4}\right) = 0$

$\Rightarrow x^4 - x^3 + 4x^3 - 4x^2 + \frac{15}{4}x^2 - \frac{15x}{4} = 0$

$\Rightarrow x^4 + 3x^3 - \frac{1}{4}x^2 - \frac{15}{4}x = 0$

Or $\Rightarrow 4x^4 + 12x^3 - x^2 - 15x = 0$

2. If α, β, γ are the roots of $4x^3 - 6x^2 + 7x + 3 = 0,$ then find the value of $\alpha\beta + \beta\gamma + \gamma\alpha.$

Sol: α, β, γ are the roots of $4x^3 - 6x^2 + 7x + 3 = 0$

$$\alpha + \beta + \gamma = -\frac{a_1}{a_0} = \frac{6}{4}$$

$$\alpha\beta + \beta\gamma + \gamma\alpha = \frac{a_2}{a_0} = \frac{7}{4}$$

$$\alpha\beta\gamma = -\frac{a_3}{a_0} = -\frac{3}{4}$$

$$\therefore \alpha\beta + \beta\gamma + \gamma\alpha = \frac{7}{4}$$

3. If $1, 1, \alpha$ are the roots of $x^3 - 6x^2 + 9x - 4 = 0,$ then find α

Sol: $1, 1, \alpha$ are roots of $x^3 - 6x^2 - 9x - 4 = 0$

Sum $= 1 + 1 + \alpha = 6$

$\alpha = 6 - 2 = 4$

4. If $-1, 2$ and α are the roots of $2x^3 + x^2 - 7x - 6 = 0,$ then find α

Sol: $-1, 2, \alpha$ are roots of $2x^3 + x^2 - 7x - 6 = 0$

Sum $= -1 + 2 + \alpha = -\frac{1}{2}$

$$\alpha = -\frac{1}{2} - 1 = -\frac{3}{2}$$

5. **If** $1, -2$ **and 3 are roots of** $x^3 - 2x^2 + ax + 6 = 0,$ **then find a.**

Sol: $1, -2$ and 3 are roots of

$x^3 - 2x^2 + ax + 6 = 0$

$\Rightarrow 1(-2) + (-2)3 + 3.1 = a$

i.e., $a = -2 - 6 + 3 = -5$

6). **If the product of the roots of** $4x^3 + 16x^2 - 9x - a = 0$ **is 9, then find a.**

Sol: α, β, γ are the roots of $4x^3 + 16x^2 - 9x - a = 0$

$\alpha\beta\gamma = \frac{a}{4} = 9 \Rightarrow a = 36$

7. **Find the values of** S_1, S_2, S_3 **and** S_4 **for each of the following equations.**

(i) $x^4 - 16x^3 + 86x^2 - 176x + 105 = 0$

Sol: Given equation is

$x^4 - 16x^3 + 86x^2 - 176x + 105 = 0$

We know that $s_1 = -\frac{a_1}{a_0} = \frac{16}{1} = 16$

$s_2 = \frac{a_2}{a_0} = \frac{86}{1} = 86$

$s_3 = -\frac{a_3}{a_0} = \frac{176}{1} = 176$

$s_4 = \frac{a_4}{a_0} = \frac{105}{1} = 105$

(ii) $8x^4 - 2x^3 - 27x^2 + 6x + 9 = 0$

Sol: Equation is $8x^4 - 2x^3 - 27x^2 + 6x + 9 = 0$

$$s_1 = -\frac{a_1}{a_0} = \frac{2}{8} = \frac{1}{4}$$

$$s_2 = \frac{a_2}{a_0} = \frac{27}{8}$$

$$s_3 = -\frac{a_3}{a_0} = -\frac{6}{8} = -\frac{3}{4}$$

$$s_4 = \frac{a_4}{a_0} = \frac{9}{8}$$

8. **Solve** $x^3 - 3x^2 - 16x + 48 = 0,$ **given that the sum of two roots is zero.**

Sol: Let α, β, γ are the roots of

$x^3 - 3x^2 - 16x + 48 = 0$

$\alpha + \beta + \gamma = 3$

Given $\alpha + \beta = 0$ ($\because$ Sum of two roots is zero)

$\therefore \gamma = 3$

i.e. $x - 3$ is a factor of

$x^3 - 3x^2 - 16x + 48 = 0$

3	1	-3	-16	48
	-	3	0	-48
	1	0	-16	0

$x^2 - 16 = 0 \Rightarrow x^2 = 16$

$\Rightarrow x = \pm 4$

$\therefore$ The roots are $-4, 4, 3$

9. Find the condition that $x^3 - px^2 + qx - r = 0$ may have the sum of its roots zero.

Sol: Let α, β, γ be the roots of $x^3 - px^2 + qx - r = 0$

$\alpha + \beta + \gamma = p$ (1)

$\alpha\beta + \beta\alpha + \gamma\alpha = q$ (2)

$\alpha\beta\gamma$ (3)

Given $\alpha + \beta = 0$

($\because$ Sum of two roots is zero)

From (1), $\gamma = p$

$\because \gamma$ is a root of $x^3 - px^2 + qx - r = 0$

$\gamma^3 - p\gamma^2 + q\gamma - r = 0$

But $\gamma = p$

$\Rightarrow p^3 - p\left(p^2\right) + q\left(p\right) - r = 0$

$\Rightarrow p^3 - p^3 + qp - r = 0$

$\therefore qp = r$ is the required condition.

10. Given that the roots of $x^3+3px^2+3px+r=0$ are in

(i) A.P., show that $2p^2-3qp+r=0$

(ii) G.P., show that $p^3r=q^3$

(iii) H.P., show that $2q^3=r(3pq-r)$

Sol: Given equation is $x^3+3px^2+3px+r=0$

(i) The roots are in A.P.

Suppose $a-d, a, a+d = -3p$

$3a=-3p \Rightarrow a=-p \qquad (1)$

$\because 'a'\ x^3+3px^2+3qx+r=0$

$\Rightarrow a^3+3pa^2+3qa+r=0$

But $a=-p$

$\Rightarrow -p^3+3p(-p)^2+3q(-p)+r=0$

$\Rightarrow 2p^3-3pq+r=0$ is the required condition

(ii) The roots are in G.P.

Suppose the roots be $\frac{a}{R}, a, aR$

Given $\left(\frac{a}{R}\right)(a)(aR)=-r$

$\Rightarrow a^3=-r$

$\Rightarrow a=(-r)^{1/3}$

$\because$ 'a' is a root of $x^3+3px^2+3qx+r=0$

$$\Rightarrow\left(-r^{1/3}\right)^3+3p\left(-r^{1/3}\right)^2+3q\left(-r^{1/3}\right)+r=0$$

$$\Rightarrow -r+3pr^{2/3}-3qr^{1/3}+r=0$$

$$pr^{2/3}=qr^{1/3}$$

$$\Rightarrow pr^{1/3}=q$$

$\Rightarrow p^3r=q$ is the required condition

(iii) The roots of $x^3+3px^2+3qx+r=0$ $\quad(1)$ are in H.P.

Let $y=\frac{1}{x}$ so that $\frac{1}{y^3}+\frac{3p}{y^2}+\frac{3q}{y}+r=0$ $\quad(2)$ are in A.P.

Suppose $a-d,a,a+d$ be the roots of (2)

Sum = $a-d,a,a+d=-\frac{3q}{r}$

$$3a=-\frac{3q}{r}$$

$$a=-\frac{q}{r} \qquad (1)$$

$\because$ 'a' is root of $ry^3+3qy^2+3py+1=0$

$$\Rightarrow ra^3+3qa^2+3pa+1=0$$

But $a=-\frac{q}{r}$

$$\Rightarrow r\left(-\frac{q}{r}\right)^3+3q\left(-\frac{q}{r}\right)^2+3p\left(-\frac{q}{r}\right)+1=0$$

$$\frac{-q^3}{r^2}+\frac{3q^3}{r^2}-\frac{3pq}{r}+1=0$$

$$\Rightarrow -q^3+3q^3-3pqr+r^2=0$$

$\Rightarrow 2q^3 = r(3pq - r)$ is the required condition.

11. Find the condition that $x^3 - px^2 + qx - r = 0$ may have the roots in G.P.

Sol: Let $\frac{a}{R}, a, aR$ be the roots of $x^3 - px^2 + qx - r = 0$

Then product of the roots

$= \frac{a}{R}, a, aR = a^3 = r$

$\Rightarrow a = r^{1/3}$

$\because$ a is a root of $x^3 - px^2 + qx - r = 0$

$\Rightarrow a^3 - pa^2 + qa - r = 0$

But $a = r^{1/3}$

$\Rightarrow \left(r^{1/3}\right)^3 - pa^2 + qa - r = 0$

But $a = r^{1/3}$

$\Rightarrow \left(r^{1/3}\right)^3 - p\left(r^{1/3}\right)^2 + q\left(r^{1/3}\right) - r = 0$

$\Rightarrow r - p.r^{2/3} + q.r^{1/3} - r = 0$

$\Rightarrow p.r^{2/3} = q\,r^{1/3}$

By cubing on both sides

$\Rightarrow p^3 r^2 = q^3 r$

$\Rightarrow p^3 r = q^3$ is the required condition

12. Form the polynomial equation whose roots are

(i) $2+3i, 2-3i, 1+i, 1-i$

Sol: The required equation is

$$\left[x-(2-3i)\right]\left[x-(2-3i)\right]$$

$$\left[x-(1+i)\right]\left[x-(1-i)\right]=0$$

$$\Rightarrow\left[(x-2)^2-9i^2\right]\left[(x-1)^2-i^2\right]=0$$

$$\Rightarrow\left(x^2-4x+4-9\right)\left(x^2-2x+1+1\right)=0$$

$$\Rightarrow\left(x^2-4x+13\right)\left(x^2-2x+2\right)=0$$

$$\Rightarrow x^3-4x^3+13x^2-2x^3+8x^2-26x+2x^2-8x+26=0$$

$$\Rightarrow x^4-6x^3+23x^2-34x+26=0$$

(ii) 3, 2, 1 + i, 1 – i

Sol: Required equation is

$$(x-3)(x-2)\left[x-(1+i)\right]\left[x-(1-i)\right]=0$$

$$\Rightarrow\left(x^2-5x+6\right)\left[(x-1)-i\right]\left[(x-1)+i\right]=0$$

$$\Rightarrow\left(x^2-5x+6\right)\left[(x-1)^2-i^2\right]=0$$

$$\Rightarrow\left(x^2-5x+6\right)\left(x^2-2x+1+1\right)=0$$

$$\Rightarrow\left(x^2-5x+6\right)\left(x^2-2x+2\right)=0$$

$$\Rightarrow x^4-5x^3+6x^2-2x^3+10x^2-12x+2x^2-10x+12=0$$

$$\Rightarrow x^4-7x^3+18x^2-22x+12=0$$

(iii) $1+i, 1-i, -1+i, -1-i$

Sol: Required equation is $\left[x-(1+i)\right]\left[x-(1-i)\right]$

$\left[x-(-1+i)\right]\left[x-(-1-i)\right]=0$

$\Rightarrow\left[(x-1)-i\right]\left[(x-1)+i\right]$

$\left[(x+1)-i\right]\left[(x+1)+I\right] = 0$

$\Rightarrow\left[(x-1)^2-i^2\right]\left[(x+1)^2-i^2\right]=0$

$\Rightarrow\left(x^2-2x+1+1\right)\left(x^2+2x+1+1\right)=0$

$\Rightarrow\left(x^2-2x+2\right)\left(x^2+2x+2\right)=0$

$\Rightarrow x^4-2x^3+2x^2+2x^3-4x^2+4x+2x^2-4x+4=0$

$\Rightarrow x^4+4=0$

(iv) $1+i, 1-i, 1+i, 1-i$

Sol: Required equation is $\left[x-(1+i)\right]\left[x-(1-i)\right]$

$\left[x-(1+i)\right]\left[x-(1-i)\right]=0$

$\Rightarrow\left[(x-1)-i\right]^2\left[(x-1)+i\right]^2=0$

$\Rightarrow\left[(x-1)^2-i^2\right]=0$

$\Rightarrow\left(x^2-2x+1+\right)^2=0$

$\Rightarrow x^4+4x^2+4-4x^3+4x^2-8x=0$

$\Rightarrow x^4-4x^3+8x^2-8x+4=0$

14. Form the polynomial equation with rational coefficients whose roots are

(i) $4\sqrt{3}, 5+2i$

Sol: For the polynomial equation with rational coefficients. The roots are conjugate surds and conjugate complex numbers

(i) $4\sqrt{3}, 5+2i$

Let $\alpha = 4\sqrt{3}$ *then* $\beta = -4\sqrt{3}$, and

$\gamma + 5 + 2i$ then $\delta = 5 - 2i$

$\alpha, \beta, \gamma, \delta$ are the roots

$\alpha + \beta = 0, \alpha\beta = -48$

$\gamma + \delta = 10, \gamma\delta = 25 + 4 = 29$

The required equations is,

$\left[x^2 - (\alpha+\beta)x + \alpha\beta\right]\left[x^2 = (\gamma+\delta)x + \gamma\delta\right] = 0$

$\Rightarrow (x^2 - 48)(x^2 - 10x + 29) = 0$

$\Rightarrow x^4 - 10x^3 + 29x^2 - 48x^2 + 480x - 132 = 0$

i.e., $x^4 - 10x^3 - 19x^2 + 480x - 1932 = 0$

(ii) $1+5i, 5-i$

Sol: For the polynomial equation with rational coefficients. The roots are conjugate surds and conjugate complex numbers.

Let $\alpha = 1 + 5i$ *then* $\beta = 1 = 5i$,

And $\gamma = 5 + i$ *then* $\delta = 5 - i$,

$\alpha + \beta = 2, \alpha\beta = 26$

$\gamma + \delta = 10, \gamma\delta = 26$

The required equation is

$$\left[x^2-(\alpha+\beta)x+\alpha\beta\right]\left[x^2-(\gamma+\delta)x+\gamma\delta\right]=0$$

$$\Rightarrow\left(x^2-2x+26\right)\left(x^2-10x+26\right)=0$$

$$\Rightarrow x^4-12x^3+72x^2-312x+676=0$$

(iii) $i-\sqrt{5}$

Sol: For the polynomial equation with rational coefficients. The roots are conjugate surds and conjugate complex numbers.

Let $\alpha=i-\sqrt{5},\beta=i+\sqrt{5}$

$\gamma=i-\sqrt{5},\delta=-i+\sqrt{5}$ are the roots

$\alpha+\beta=2i,\alpha\beta=-6$

$\gamma+\delta=-2i,\gamma\delta=-6$

The required equation is

$$\left[x^2-(\alpha+\beta)x+\alpha\beta\right]\left[x^2-(\gamma+\delta)x+\gamma\delta\right]=0$$

$$\Rightarrow\left(x^2-2ix-6\right)\left(x^2+2ix-6\right)=0$$

$$\Rightarrow\left[\left(x^2-6\right)-2ix\right]\left[\left(x^2-6\right)+2ix\right]=0$$

$$\Rightarrow\left(x^2-6\right)^2+4x^2=0$$

$$\Rightarrow x^4+36-12x^2+4x^2=0$$

$$\Rightarrow x^4-8x^2+36=0$$

(iv) $-\sqrt{3}+i\sqrt{2}$

Sol: Let $\alpha=-\sqrt{3}+i\sqrt{2}, \beta=-\sqrt{3}-i\sqrt{2}$

$\gamma=\sqrt{3}-i\sqrt{2}, \delta=\sqrt{3}+i\sqrt{2}$

$\alpha+\beta=-2\sqrt{3}, \alpha\beta=\left(-\sqrt{3}\right)^2-\left(i\sqrt{2}\right)^2$

$=3=i^2\left(2\right)=5$

$\gamma+\delta=2\sqrt{3}, \gamma\delta=5$

The required equation is

$\left(x^2-\left(\alpha+\beta\right)x+\alpha\beta\right)\left(x^2-\left(\gamma+\delta\right)x+\gamma\delta\right)=0$

$\Rightarrow\left(x^2+2\sqrt{3}x+5\right)\left(x^2-2\sqrt{3}x+5\right)=0$

$\Rightarrow\left(x^2+5\right)^2-\left(2\sqrt{3}x\right)^2=0$

$\Rightarrow x^4+25+10x^2-12x^2=0$

$\Rightarrow x^4-2x^2+25=0$

15. Find the algebraic equation whose roots are 3 times the roots of $x^3+2x^2-4x+1=0$

Sol: Given equation is $f\left(x\right)=x^3+2x^2-4x+1=0$

We require an equation whose roots are 3 times the roots of $f\left(x\right)=0$

i.e., Required equation is $f\left(\frac{x}{3}\right)=0$

$\left(\frac{x}{3}\right)^3+2\left(\frac{x}{3}\right)^2-\frac{4x}{3}+1=0$

$$\frac{x^3}{27}+\frac{2}{9}x^2-\frac{4}{3}x+1=0$$

Multiplying with 27, required equation is $x^3 - 36x+27=0$

16. Find the algebraic equation whose roots are 2 times the roots of

$$x^5-2x^4+3x^3-2x^2+4x+3=0$$

Sol: Given equation is

$$f(x)=x^5-2x^4+3x^3-2x^2+4x+3=0$$

We require an equation whose roots are 2 times the roots of $f(x)=0$

Required equation is $f\left(\frac{x}{2}\right)=0$

$$\Rightarrow\left(\frac{x}{2}\right)^5-2\left(\frac{x}{2}\right)^4+3\left(\frac{x}{2}\right)-2\left(\frac{x}{2}\right)^2+4\left(\frac{x}{2}\right)+3=0$$

$$\Rightarrow\frac{x^5}{32}-2\frac{x^4}{16}+3.\frac{x^3}{8}-2.\frac{x^2}{4}+4.\frac{x}{2}+3=0$$

Multiplying with 32, required equation is

$$\Rightarrow x^2-4x^4+12x^3-16x^2+64x+96=0$$

17. Find the transformed equation whose roots are the negative of the roots of
$x^4+5x^3+11x+3=0$

Sol: Given $f(x)=x^4+5x^3+11x+3=0$

We want an equation whose roots are

$$-\alpha_1,-\alpha_2,-\alpha_3,-\alpha_4$$

Required equation $f(-x)=0$

$\Rightarrow (-x)^4+5(-x)^3+11(-x)+3=0$

$\Rightarrow x^4-5x^3-11x+3=0$

18. Find the transformed equation whose roots are the negative of the roots of

$x^7+3x^5+x^3-x^2+7x+2=0$

Sol: Given $f(x)=x^7+3x^5+x^3-x^2+7x+2=0$

We want an equation whose roots are

$-\alpha_1,-\alpha_2,.......-\alpha_n$

Required equation is $f(-x)=0$

$(-x)^7+3(-x)^5+(-x)^3-(-x)^2+7(-x)+2=0$

$\Rightarrow -x^7+-3x^3-x^2-7x+2=0$

$\Rightarrow x^7+3x^5+x^3+x^2+7x-2=0$

19. Find the polynomial equation whose roots are the reciprocals of the roots of

$x^4-3x^3+7x^2+5x-2=0$

Sol: Given equation is $f(x)$

$x^5+11x^4+x^3+4x^2-13x+6+0$

Required equation is $f\left(\frac{1}{x}\right)=0$

$\frac{1}{x^5}+\frac{11}{x^4}+\frac{1}{x^3}+\frac{4}{x^2}-\frac{13}{x}+6=0$

Multiplying by x^5

$\Rightarrow 1+11x+x^2+4x^3-13x^4+6x^5=0$

i.e., $6x^5-13x^4+4x^3+x^2+11x+1=0$

20. Form the polynomial equation of degree 3 whose roots are 2, 3 and 6.

Sol: The required polynomial equation is

$(x-\ )(x-3)(x-6)=0$

$\Rightarrow x^3-11x^2+36x-36=0$

21. Find the relation between the roots and the coefficient of the cubic equation

$3x^2-10x^2+7x+10=0$

Sol; $3x^3-10x^2+7x+10=0$ --------- (1)

On comparing (1) with

$ax^3+bx^2+cx+d=0$

We have

$$\sum \alpha=\frac{-b}{a}=\frac{-(-10)}{3}=\frac{10}{3}$$

$$\sum \alpha\beta=\frac{c}{a}=\frac{7}{3}$$

And $\alpha\beta\gamma=\dfrac{-d}{a}=\dfrac{-10}{3}$

22. Write down the relations between the roots and the coefficients of the bi-quadratic equation $x^4-2x^3+4x^2+6x-21=0$

Sol: Given equation is

$x^4-2x^3+4x^2+6x-21=0$ -------- (1)

On comparing (1) with

$ax^3+bx^3+cx^2+dx+c=0$

We have

$a=1, b=-2, c=4, d=6, e=-21$

$$\sum \alpha=\frac{-b}{a}=\frac{-(-2)}{1}=2$$

$$\sum \alpha\beta=\frac{c}{a}=\frac{4}{1}=4$$

$$\sum \alpha\beta\gamma=\frac{-d}{a}=-6$$

And $\alpha\beta\gamma\delta=\frac{e}{a}=-21$

23. If 1, 2, 3 and 4 are the roots of $x^4+ax^3+bx^2+cx+d=0$**, then find the values of** $a, b, c \text{ and } d$

Sol: Given that the roots of the given equation are 1, 2, 3 and 4 then

$x^4+ax^3+bx^2+cx+d$

$=(x-1)(x-2)(x-3)(x-4)=0$

$=(x^4-10x^3+35x^2-50x+24=0)$

On equating the coefficients of like of powers of x, we obtain

$a = -10, b = 35, c = -50, d = 24$

24. If a, b, c are the roots of $x^3 - px^2 + qx - r = 0$ and $r \neq 0$, then find $\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}$

Sol: Given that a, b, c are the roots of

$x^3 - px^2 + qx - r = 0$, then

$a + b + c = p, ab + bc + ca = q, abc = r$

Now $\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} = \frac{b^2c^2 + a^2c^2 + a^2b^2}{a^2b^2c^2}$

$= \frac{(ab + bc + ca)^2 - 2abc(a + b + c)}{a^2b^2c^2}$

$= \frac{q^2 - 2pr}{r^2}$

25. If α, β, γ are roots of the equations $x^3 - 10x^2 + 6x - 8 = 0$ find $\alpha^2 + \beta^2 + \gamma^2$

Sol: Given α, β, γ are the roots of the equation $x^3 - 10x^2 + 6x - 8 = 0$

$\alpha + \beta + \gamma = 10, \alpha\beta + \beta\gamma + \gamma\alpha = 6, \alpha\beta\gamma = 8$

Now $\alpha^2 + \beta^2 + \gamma^2 = (\alpha + \beta + \gamma)^2 = -2(\alpha\beta + \beta\gamma + \gamma\alpha)$

$= 100 - 12$

$\therefore \alpha^2 + \beta^2 + \gamma^2 = 88$

26. Find the sum of the squares and the sum of the cubes of the roots of the equation

$x^3 - px^2 + qx - r = 0$

Sol: Let α, β, γ be the roots of the given equation then $\alpha + \beta + \gamma = p, \alpha\beta + \beta\gamma + \gamma\alpha = q, \alpha\beta\gamma = r$

Sum of the squares of the roots is $\alpha^2 + \beta^2 + \gamma^2 = (\alpha + \beta + \gamma)^2 - 2(\alpha\beta + \beta\gamma + \gamma\alpha) = p^2 - 2q$

Sum of the cubes of the roots is

$$\alpha^3 + \beta^3 + \gamma^3 = (\alpha + \beta + \gamma)\left(\alpha^2 + \beta^2 + \gamma^2 - \alpha\beta - \beta\gamma - \gamma\alpha\right) + 3\alpha\beta\gamma$$

$$= p\left(p^2 - 2q - q\right) + 3r$$

$$= p\left(p^2 - 3q\right) + 3r$$

27. Obtain the cubic equation, whose roots are the squares of the roots of the equations,

$x^3 + p_1x^2 + p_2x + p_3 = 0$

Sol: The required equation is $f\left(\sqrt{x}\right) = 0$

$$\Rightarrow \left(\sqrt{x}\right)^3 + p_1\left(\sqrt{x}\right)^2 + p_2\left(\sqrt{x}\right) + p_3 = 0$$

$$\Rightarrow x\sqrt{x} + p_1x + p_2\left(\sqrt{x}\right) + p_3 = 0$$

$$\Rightarrow \sqrt{x}\,(x + p_2) = -(p_1x + p_3)$$

Squaring on both sides

$$\Rightarrow x(x + p_2)^2 = (p_1x + p_3)^2$$

$$\Rightarrow x\left(x^2 + p_2^2 + 2p_2x\right) = p_1^2x^2 + p_3^{\,2} + 2p_1p_3x$$

$$\Rightarrow x^3 + p_2^{\,2}x + 2p_2x^2 - p_1^{\,2}x^2 - p_3^{\,2} - 2p_1p_3\,x = 0$$

$$\therefore x^2 + \left(2p_2 - p_1^2\right)x^2 + \left(p_2^2 - 2p_1p_3\right)x - p_3^2 = 0$$

28. **Let** α, β, γ **be the roots of** $x^3 + px^2 + qx + r = 0$**. Then find the following,** $\alpha + \beta + \gamma = -p$**,**
$\alpha\beta + \beta\gamma + \gamma\alpha = q, \alpha\beta\gamma = -r$

(i) $\sum \alpha^2$

Sol: $\sum \alpha^2 = \alpha^2 + \beta^2 + \gamma^2$

$= (\alpha + \beta + \gamma)^2 - 2(\alpha\beta + \beta\gamma + \gamma\alpha)$

$= p^2 - 2q$

(ii) $\sum \frac{1}{\alpha}$

Sol: $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = \frac{\beta\gamma + \gamma\alpha + \alpha\beta}{\alpha\beta\gamma} = \frac{-q}{r}$

(iii) $\sum \alpha^3$

Sol: $\sum \alpha^3 = \alpha^3 + \beta^3 + \gamma^3$

$= (\alpha + \beta + \gamma)(\alpha^2 + \beta^2 + \gamma^2 - \alpha\beta - \gamma\alpha) + 3\alpha\beta$

$= (-p)(p^2 - 2q - q) - 3r$

$= -p(p^2 - 3q) - 3r$

$\therefore \sum \alpha^3 = -p^3 + 3pq - 3r = 3pq - p^3 - 3r$

(iv) $\sum \beta^2\gamma^2$

Sol: $\sum \beta^2\gamma^2 = \beta^2\gamma^2 + \gamma^2\alpha^2 + \alpha^2\beta^2$

$= (\beta\gamma + \gamma\alpha + \alpha\beta)^2 - 2\alpha\beta\gamma(\alpha + \beta + \gamma)$

$= q^2 - 2(-r)(-p)$

$= q^2 - 2pr$

(v) $(\alpha+\beta)(\beta+\gamma)(\gamma+\alpha)$

Sol: We know $\alpha + \beta + \gamma = -p$

$\Rightarrow \gamma + \alpha = -p - \beta$

$\therefore (\alpha+\beta)(\beta+\gamma)(\gamma+\alpha)$

$= (-p-\gamma)(-p-\alpha)(-p,-\beta)$

$= -p^3 - p^2(\alpha+\beta+\gamma) - p(\alpha\beta+\beta\gamma+\gamma\alpha) - \alpha\beta\gamma$

$= -p^3 + p^3 - pq + r = r - pq$

29. Let α, β, γ be the roots of $x^3 + ax^2 + bx + c = 0$ then find $\sum \alpha^2\beta^2\gamma^2 + \sum \alpha\beta^2$

Sol: Since α, β, γ are roots of the given equation, we have

$\alpha + \beta + \gamma = -a, \alpha\beta + \beta\gamma + \gamma\alpha = b, \alpha\beta\gamma = -c$

Now $\sum \alpha^2\beta + \sum \alpha\beta^2 = \sum \alpha\beta(\alpha+\beta)$

$= \sum \alpha\beta(-a-\gamma)$

$= -a\sum \alpha\beta - \sum \alpha\beta\gamma$

$= -a(\alpha\beta+\beta\gamma+\gamma\alpha) - 3\alpha\beta\gamma$

$= -a(+b) + 3c$

$= 3c - ab$

$\therefore \sum \alpha^2\beta + \sum \alpha\beta^2 = 3c - ab$

30. Form the polynomial equation of degree 4 whose roots we $4+\sqrt{3}, 4-\sqrt{3}, 2+i$ and $2-i$

Sol: The equation having roots $4+\sqrt{3}, 4-\sqrt{3}$ is

$x^2-8x+13=0$

The equation having roots $2+i, 2-i$ is

$x^2-4x+5=0$

The required equation is

$(x^2-8x+13)(x^2-4x+5)=0$

$\therefore x^4-12x^3+50x^2-92x+65=0$

Short Answer Questions

1. If α, β and 1 are the roots of $x^3-2x^2-5x+6=0$, then find α and β

Sol: α, β and 1 are the roots of

$x^3-2x^2-5x+6=0$

Sum $\alpha+\beta+1=2 \Rightarrow \alpha+\beta=1$

Product $=\alpha\beta=-6$

$(\alpha-\beta)^2=(\alpha+\beta)^2-4\alpha\beta$

$=1+24=25$

$\alpha-\beta=5$

$\alpha+\beta=1$

Adding $2\alpha=6 \Rightarrow \alpha=3$

$\alpha+\beta=1 \Rightarrow \beta=1-\alpha=1-3=-2$

$\therefore \alpha=3$ and $\beta=-2$

2. If α,β and γ are the roots of $x^3-2x^2+3x-4=0$, then find

(i) $\sum\alpha^2\beta^2$ **(ii)** $\sum\alpha^2\beta+\sum\alpha\beta^2$

Sol: Since α,β,γ are the roots of

$x^3-2x^2+3x-4=0$ then $\alpha+\beta+\gamma=2$

$\alpha\beta+\beta\gamma+\gamma\alpha=3$

$\alpha\beta\gamma=4$

(i) $\sum\alpha^2\beta^2=\alpha^2\beta^2+\beta^2\gamma^2+\gamma^2\alpha^2$

$=(\alpha\beta+\beta\gamma+\gamma\alpha)^2-2\alpha\beta\gamma(\alpha+\beta+\gamma)$

$=9-2.2.4=9-16=-7$

(ii) $\sum\alpha^2\beta=\alpha^2\beta+\beta^2\gamma+\gamma^2\alpha+\alpha\beta^2+\beta\gamma^2+\gamma\alpha^2$

$=(\alpha\beta+\beta\gamma+\gamma\alpha)(\alpha+\beta+\gamma)-3\alpha\beta\gamma$

$2.3-3.4=6-12=-6.$

3. If α,β and γ are the roots of $x^3+px^2+qx+r=0$, then find the following.

Sol: α,β and γ are the roots of $x^3+px^2+qx+r=0.$

$\alpha+\beta+\gamma=-p.$

$\alpha\beta+\beta\gamma+\gamma\alpha=q$

$\alpha\beta\gamma=-r$

(i) $\sum \frac{1}{\alpha^2\beta^2}$

Sol: $\sum \frac{1}{\alpha^2\beta^2} = \frac{1}{\alpha^2\beta^2} + \frac{1}{\beta^2\gamma^2} + \frac{1}{\gamma^2\alpha^2}$

$$\frac{\alpha^2+\beta^2+\gamma^2}{\alpha^2+\beta^2+\gamma^2}$$

$$= \frac{(\alpha+\beta+\gamma)^2 - 2(\alpha\beta+\beta\gamma+\gamma\alpha)}{(\alpha\beta\gamma)^2}$$

$$= \frac{(-p)^2 - 2q}{(-r)^2} = \frac{p^2-2q}{r^2}$$

(ii) $\frac{\beta^2+\gamma^2}{\beta\gamma} + \frac{\gamma^2+\alpha^2}{\gamma\alpha} + \frac{\alpha^2+\beta^2}{\alpha\beta}$ or $\sum \frac{\beta^2+\gamma^2}{\beta\gamma}$

Sol: $\sum \frac{\beta^2+\gamma^2}{\beta\gamma} = \frac{\beta^2+\gamma^2}{\beta\gamma} + \frac{\gamma^2+\alpha^2}{\gamma\alpha} + \frac{\alpha^2+\beta^2}{\alpha\beta}$

$$= \frac{\alpha\beta^2+\alpha\gamma^2+\gamma^2\beta+\alpha^2\beta+\alpha^2\gamma+\beta^2\gamma}{\alpha\beta\gamma}$$

$$= \frac{(\alpha\beta+\beta\gamma+\gamma\alpha)(\alpha+\beta+\gamma)-3\alpha\beta\gamma}{\alpha\beta\gamma}$$

$$= \frac{-pq+3r}{-r} = \frac{pq-3r}{r}$$

(iii) $(\beta+\gamma-3\alpha)(\gamma+\alpha-3\beta)(\alpha+\beta-3\gamma)$

Sol: $(\beta+\gamma-3\alpha)(\gamma+\alpha-3\beta)(\alpha+\beta-3\gamma)$

$$= (\alpha+\beta+\gamma-4\alpha)(\alpha+\beta+\gamma-4\beta)(\alpha+\beta+\gamma-4\gamma)$$

$$= (-p-4\alpha)(-p-4\beta)(-p-4\gamma)$$

$$= -(p+4\alpha)(p+4\beta)(p+4\gamma)$$

$$= -(p^3+4p^2)(\alpha+\beta+\gamma)+16p(\alpha\beta+\beta\gamma+\gamma\alpha)+(64\alpha\beta\gamma)$$

$$= -(p^3-4p^3+16pq-64r)$$

$$= 3p^3-16pq+64r$$

(iv) $\sum\alpha^3\beta^3$

Sol: $\sum\alpha^3\beta^3 = \alpha^3\beta^3+\beta^3\gamma^3+\gamma^3\alpha^3$

$$(\alpha\beta+\beta\gamma+\gamma\alpha)^2 = \alpha^2\beta^2+\beta^2\gamma^2+\gamma^2\alpha^2+2\alpha\beta\gamma(\alpha+\beta+\gamma)$$

$$q^2 = \alpha^2\beta^2+\beta^2\gamma^2+\gamma^2\alpha^2+2pr$$

$$\alpha^2\beta^2+\beta^2\gamma^2+\gamma^2\alpha^2 = q^2-2pr$$

$$\therefore \alpha^3\beta^3+\beta^3\gamma^3+\gamma^3\alpha^3 = (\alpha^2\beta^2+\beta^2\gamma^2+\gamma^2\alpha^2)(\alpha\beta+\beta\gamma+\gamma\alpha)-\alpha\beta\gamma\sum\alpha^2\beta$$

$$= (q^2-2pr).q+r\left[(\alpha\beta+\beta\gamma+\gamma\alpha)(\alpha+\beta+\gamma)-3\alpha\beta\gamma\right]$$

$$= q^3-2pqr+r(-pq+3r)$$

$$= q^3-2pqr-pqr+3r^2 = q^3-3pqr+3r^2$$

4. Solve $9x^3-15x^2+7x-1=0$, **given that two of its roots are equal.**

Sol: Suppose α,β,γ are the roots of

$$9x^3-15x^2+7x-1=0$$

$$\alpha+\beta+\gamma = \frac{15}{9} = \frac{5}{3}$$

$$\alpha\beta+\beta\gamma+\gamma\alpha=\frac{7}{9}$$

$$\alpha\beta\gamma=\frac{1}{9}$$

Given $\alpha=\beta$ ($\because$ two its roots are equal)

$$2\alpha+\gamma=\frac{5}{3}\Rightarrow\gamma=\frac{5}{3}-2\alpha$$

$$\alpha^2+2\alpha\gamma=\frac{7}{9}$$

$$\Rightarrow\alpha^2+2\alpha\left(\frac{5}{3}-2\alpha\right)=\frac{7}{9}$$

$$\Rightarrow\alpha^2+\frac{2\alpha(5-6\alpha)}{3}=\frac{7}{9}$$

$$\Rightarrow 9\alpha^2+6\alpha(5-6\alpha)=7$$

$$9\alpha^2+30\alpha-36\alpha^2=7$$

$$\Rightarrow 27\alpha^2-30\alpha+7=0$$

$$\Rightarrow(3\alpha-1)(9\alpha-7)=0$$

$$\Rightarrow\alpha=\frac{1}{3}\ or\ \frac{7}{9}$$

Case (i) $\alpha=\frac{1}{3}$

$$\gamma=\frac{5}{3}-2\alpha=\frac{5}{3}-\frac{2}{3}=1$$

The roots are $\frac{1}{3},\frac{1}{3},1$

Case (ii) $\alpha = \frac{7}{9}$

$$\gamma = \frac{5}{3} - 2\alpha = \frac{5}{3} - \frac{14}{9} = \frac{1}{9}$$

$\alpha\beta\gamma = \frac{7}{9}.\frac{7}{9}.\frac{1}{9} \neq \frac{1}{9} \Rightarrow$ Does not satisfy the given equation.

The roots are $\frac{1}{3}, \frac{1}{3}, 1$

5.Given that one root of $2x^3 + 3x^2 - 8x + 3 = 0$ is double of another root, find the roots of the equation.

Sol: Suppose α, β, γ are the roots of $2x^3 + 3x^2 - 8x + 3 = 0$

$$\alpha + \beta + \gamma = -\frac{3}{2} \qquad (1)$$

$$\alpha\beta + \beta\gamma + \gamma\alpha = -4 \qquad (1)$$

$$\alpha\beta\gamma = -\frac{3}{2}$$

Given $\alpha = 2\beta$ ($\because$ one root is double the other substituting in (1)

$$3\beta + \gamma = -\frac{3}{2} \Rightarrow \gamma = -\frac{3}{2} - 3\beta \qquad (4)$$

Substituting in (2)

$$\alpha\beta + \gamma(\alpha + \beta) = -4$$

$$\Rightarrow 2\beta^2 + 3\beta\gamma = -4$$

$$\Rightarrow 2\beta^2 + 3\beta\left(-\frac{3}{2} - 3\beta\right) = -4$$

$$\Rightarrow 2\beta^2 - \frac{3\beta(3+6\beta)}{2} = -4$$

$$\Rightarrow 4\beta^2 - 9\beta - 18\beta^2 = -8$$

$$\Rightarrow 14\beta^2 + 9\beta - 8 = 0$$

$$\Rightarrow (2\beta - 1)(7\beta + 8) = 0$$

$$\Rightarrow 2\beta - 1 = 0 \ or \ 7\beta + 8 = 0$$

$$\Rightarrow \beta = \frac{1}{2} \ or \ \beta = -\frac{8}{7}$$

Case (i) $\beta = \frac{1}{2}$

$$\alpha = 2\beta = 2 \times \frac{1}{2} = 1$$

$$\gamma = -\frac{3}{2} - 3\beta = -\frac{3}{2} - \frac{3}{2} = -3$$

$\alpha\beta\gamma = 1\left(\frac{1}{2}\right)(-3) = -\left(\frac{3}{2}\right)$ is satisfied

The roots are $1, \frac{1}{2}, -3$

Case (ii) $\beta = -\frac{8}{7}$

$$\alpha = 2\beta = -\frac{16}{7}$$

$$\gamma = -\frac{3}{2} - 3\beta = -\frac{3}{2} + \frac{48}{7} = \frac{75}{14}$$

$\alpha\beta\gamma = -\frac{3}{2}$ is not satisfied.

$\therefore$ The roots are $\frac{1}{2}, 1$ and -3

6. **Solve** $x^3 - 9x^2 + 14x + 24 = 0$ **given that two of the roots are in the ratio** $3:2$.

Sol: **Suppose** α, β, γ **are the roots of** $x^3 - 9x^2 + 14x + 24 = 0$

$\therefore \alpha + \beta + \gamma = 9 \qquad (1)$

$\therefore \alpha\beta + \beta\gamma + \gamma\alpha = 14 \qquad (1)$

$\alpha\beta\gamma = -24 \qquad (1)$

$\because$ two roots are in the ratio $3:2$

Let $\alpha : \beta = 3:2 \Rightarrow \beta = \frac{2\alpha}{3}$

Substituting in (1)

$\frac{5\alpha}{3} + \gamma = 9 \Rightarrow \gamma = 9 - \frac{5\alpha}{3} \qquad (4)$

Substituting in (2)

$\Rightarrow \frac{2}{3}\alpha^2 + \gamma(\alpha + \beta) = 14$

$\Rightarrow \frac{2}{3}\alpha^2 + \left(9 - \frac{5\alpha}{3}\right) . \frac{5\alpha}{3} = 14$

$\Rightarrow 2\alpha^2 + 5\alpha\left(9 - \frac{5\alpha}{3}\right) = 42$

$\Rightarrow 6\alpha^2 + 135\alpha - 25\alpha^2 = 126$

$\Rightarrow 19\alpha^2 - 135\alpha + 126 = 0$

$\Rightarrow 19\alpha^2 - 114\alpha - 21\alpha + 126 = 0$

$\Rightarrow 19\alpha(\alpha-6)-21(\alpha-6)=0$

$\Rightarrow (19\alpha-21)(\alpha-6)=0$

$\Rightarrow 19\alpha-21=0$ *or* $\alpha-6=0$

$\alpha=\frac{21}{19}$ *or* $\alpha=6$

Case (i) $\alpha=6$

$\beta=\frac{2\alpha}{3}=\frac{2}{3}.6=4$

$\gamma=9-\frac{5\alpha}{3}=9-\frac{5}{3}.6=9-10=-1$

$\alpha=6, \beta=4, \gamma=-1$ Satisfy $\alpha\beta\gamma=-24$

$\therefore$ The roots are $6, 4, -1$

Case (ii) $\alpha=\frac{21}{19}$

$\beta=\frac{2}{3}\times\frac{21}{19}=\frac{14}{19}$

$\gamma=9-\frac{5\alpha}{3}=9-\frac{5}{3}.\frac{21}{19}$

$=9-\frac{35}{19}=\frac{171-35}{19}=\frac{136}{19}$

These values do not satisfy $\alpha\beta\gamma=-24$

$\therefore$ The roots are $6, 4, -1$

7. Solve the following equations, given that the roots of each are in A.P.

(i) $8x^3-36x^2-18x+81=0$

Sol: Given the roots of $8x^3-36x^2-18x+81=0$ are in A.P.

Let the roots be $a-d, a, a+d$

Sum of the roots = $a-d, a, a+d$

$$=\frac{36}{8}=\frac{9}{2}$$

$$i.e., 3a=\frac{9}{2}\Rightarrow a=\frac{3}{2}$$

$\therefore\left(x-\frac{3}{2}\right)$ is a factor of $8x^3-36x^2-14x+81=0$

3/2	8	-36	-18	81
	-	12	-36	-81
	8	-24	-54	0

$$\Rightarrow 8x^2-24x-54=0$$

$$\Rightarrow 4x^2-12x-27=0$$

$$\Rightarrow 4x^2-18x+6x-27=0$$

$$\Rightarrow 2x(2x-9)+3(2x-9)=0$$

$$\Rightarrow (2x+3)(2x-9)=0$$

$$\Rightarrow x=-\frac{3}{2},\frac{9}{2}$$

The roots are $-\frac{3}{2},\frac{3}{2},\frac{9}{2}$

(ii) $x^3 - 3x^2 - 6x + 8 = 0$

Sol: The roots of $x^3 - 3x^2 - 6x + 8 = 0$ are in A.P.

Suppose $a-d, a, a+d$ be the roots

Sum = $a - d + a + a + d = 3$

$3a = 3$

$\Rightarrow a = 1$

$\therefore (x-1)$ is a factor of $x^3 - 3x^2 - 6x + 8 = 0$

1	1	-3	-6	8
	-	1	-2	-8
	1	-2	-8	0

$\Rightarrow x^2 - 2x - 8 = 0$

$\Rightarrow x^2 - 4x + 2x - 8 = 0$

$\Rightarrow x(x-4) + 2(x-4) = 0$

$\Rightarrow (x-4)(x+2) = 0.$

$\Rightarrow x = 4, -2$

$\therefore$ The roots are $-2, 1, 4.$

8. Solve the following equations, given that the roots each are in G.P.

(i) $3x^2 - 26x^2 + 52x - 24 = 0$

Sol: Given equation is $3x^3 - 26x^2 + 52x - 24 = 0$

The roots are in G.P.

Suppose $\frac{a}{r}, a, ar$ are the roots

Product $=\frac{a}{r}.a.ar$ are the roots.

Product $\frac{a}{r}.a.ar=\left(-\frac{24}{3}\right)$

$a^3=8\Rightarrow a=2$

$\therefore (x-2)$ is a factor of $3x^3-26x^2+52x-24$

2	3	-26	52	-24
	0	6	-40	-24
	3	-20	12	0

$$\begin{aligned} H\text{int}: & 3\times 12 \\ & =3\times 6\times 2 \\ & =(18)(-2) \end{aligned}$$

$\Rightarrow 3x^2-20x+12=0$

$\Rightarrow 3x^2-18x-2x+12=0$

$\Rightarrow 3x(x-6)-2(x-6)=0$

$\Rightarrow (3x-2)(x-6)=0$

$\Rightarrow x=\frac{2}{3},6$

$\therefore$ The roots are $\frac{2}{3},2,6.$

(ii) $54x^3-39x^2-26x+16=0$

Sol: Given equation is $54x^2-39x^2-26x+16=0$

The roots are in G.P.

Suppose $\frac{a}{r},a,ar$ be the roots

Product $= \frac{a}{r}, a, ar = -\frac{16}{54}$

$a^3 = \frac{-8}{27} = \left(-\frac{2}{3}\right)^3 \Rightarrow a = -\frac{2}{3}$

$\therefore x + \frac{2}{3}$ is a factor of $54x^3 - 39x^2 + 26x + 16$

-2/3	54	-39	-26	16
	0	-36	50	-16
	54	-36	24	0

$$\begin{aligned} H\text{int}: & 18 \times 8 \\ & = 9 \times 2 \times 8 \\ & = (-9)(-16) \end{aligned}$$

$\Rightarrow 54x^2 - 75x + 24 = 0$

$\Rightarrow 18x^2 - 25x + 8 = 0$

$\Rightarrow 18x^2 - 9x - 16x + 8 = 0$

$\Rightarrow 9x(2x-1) - 8(2x-1) = 0$

$\Rightarrow (9x-8)(2x-1) = 0$

$\Rightarrow x = \frac{8}{9}, \frac{1}{2}$

$\therefore$ The roots are $\frac{8}{9}, -\frac{2}{3}, \frac{1}{2}$

9. Solve the following equations, given that the roots of each are in H.P.

(i) $6x^3 - 11x^2 + 6x - 1 = 0$

Sol: Given equation is $6x^3 - 11x^2 + 6x - 1 = 0$ $\quad ---(1)$

Put $y = \frac{1}{x}$ so that $\frac{6}{y^3} - \frac{11}{y^2} + \frac{6}{y} - 1 = 0$

$$6-11y+6y^2-y^3=0$$

$$y^3-6y^2+11y-6=0 \qquad ----(2)$$

Roots of (1) are in H.P.

$\Rightarrow$ Roots of (2) are in A.P.

Let $a-d, a, a+d$ be the roots of (2)

Sum = $a-d+a+a+d=6$

$3a=6 \Rightarrow a=2$

Product = $a\left(a^2-d^2\right)=6$

$$2\left(4-d^2\right)=6$$

$$4-d^2=3$$

$$\Rightarrow d^2=1, d=1$$

$a-d=2-1=1, a=2, a+d=2+1=3$

The roots of (2) are 1, 2, 3

The roots of (1) are $1, \frac{1}{2}, \frac{1}{3}$

(ii) $15x^3-23x^2+9x-1=0$

Sol: Given equation is $15x^3-23x^2+9x-1=0 \qquad ----(1)$

Put $y=\frac{1}{x}$ so that $\frac{15}{y^3}-\frac{23}{y^2}+\frac{9}{y}-1=0$

$15-23y+9y^2-y^3=0$

(or) $y^3-9y^2+23y-15=0 \qquad ---(2)$

Roots of (i) are in H.P. So that roots of (2) are in A.P.

Let $a-d,\ a,\ a+d$ be the roots of (2)

Sum $=a-d+a+a+d=9$

$3a=9$

$a=3$

Product $=a\left(a^2-d^2\right)=15$

$3\left(9-d^2\right)=15$

$9-d^2=5$

$d^2=4\Rightarrow d=2$

$a-d=3-2=1,\ a=3$

$a+d=3+2=5$

Roots of (2) are 1, 3, 5

Hence roots of (1) are $1,\frac{1}{3},\frac{1}{5}$

10. Solve the following equations, given that they have multiple roots.

(i) $x^4-6x^3+13x^2-24x+36=0$ **(ii)** $3x^4+16x^3+24x^2-16=0$

Sol: (i) Let $f(x)=x^4-6x^3+13x^2-24x+36$

$\Rightarrow f'(x)=4x^3-18x^2+26x-24$

$\Rightarrow f'(3)=4(27)-18(9)+26(3)-24$

$=108-162+78-24=0$

$f(3)=81-162+117-72+36=0$

$\therefore x-3$ is a factor of $f'(x)$ and $f(x)$

$\therefore$ 3 is the repeated root of $f(x)$

3	1	−6	13	−24	36
	–	3	−9	12	−36
3	1	−3	4	−12	0
	–	3	0	12	
	1	0	4	0	

$x^2+4=0 \Rightarrow x=\pm 2i$

$\therefore$ The roots of the given equation are $3, 3, \pm 2i$

(ii) $3x^4+16x^3+24x^2-16=0$

Sol: Let $f(x)=3x^4+16x^3+24x^2-16$

$f'(x)=12x^3+48x^2+48x$

$=12x\left(x^2+4x+4\right)$

$12x(x+2)^2$

$f'(-2)=0$

$f(-2)=3(16)+16(-8)+24(4)-16=0$

$\therefore x+2$ is a factor of $f'(x)$ *and* $f(x)$

$\therefore -2$ is a multiple root of $f(x)=0$

	3	16	24	0	−16
−2	–	−6	−20	−8	16
	3	10	4	−8	0
−2	–	−6	−8	8	
	3	4	−4	0	

$3x^3+4x-4=0 \Rightarrow 3x^2+6x-2x-4=0$

$\Rightarrow 3x(x+2)-2(x+2)=0$

$\Rightarrow (x+2)(3x-2)=0$

$\Rightarrow x=-2,\frac{2}{3}$

$\therefore$ The roots of the given equation are

$-2,-2,-2,\frac{2}{3}$

10. Solve the equation $x^4+2x^3-5x^2+6x+2=0$ given that $1+i$ is one of its roots

Sol: Let $1+i$ is one root $\Rightarrow 1-i$ is another root

The equation having roots

$1 \pm i$ *is* $x^2-2x+2=0$

$\therefore x^2-2x+2$ is a factor is

$x^4+2x^3-5x^2+6x+2=0$

	1	2	−5	6	2
2	–	2	8	2	–
−2	–	–	−2	−8	−2
	1	4	1	0	0

$x^2+4x+1=0 \Rightarrow$

$$x=\frac{-4\pm\sqrt{16-4}}{2}=\frac{-4\pm2\sqrt{3}}{2}$$

$x=-2\pm\sqrt{3}$

$\therefore$ The roots of the given equations are

$1 \pm i, -2+\sqrt{3}$

11. Solve the equation $3x^3 - 4x^2 + x + 88 = 0$ which has $2-\sqrt{-7}$ as a roots

Sol: Let $2-\sqrt{-7}$ (*i.e.,*) $2-\sqrt{7}i$ is one root $\Rightarrow 2+\sqrt{7}i$ is another root. The equation having roots

$2 \pm \sqrt{7}$ is $x^2 - 4x + 11 = 0$

$\therefore x^2 - 4x + 11$ is a factor of the given equation

	3	– 4	1	88
4	–	12	32	–
–11	–	–	– 33	– 88
	3	8	0	0

$3x + 8 = 0 \Rightarrow x = -\dfrac{8}{3}$

$\therefore$ The roots of the given equation are

$2 \pm \sqrt{7}, i - \dfrac{8}{3}$

12. Solve $x^3 - 4x^2 + 8x + 35 = 0$, given that $2 + i\sqrt{3}$ is a root

Sol: Let $2+i\sqrt{3}$ is one root $\Rightarrow 2 - i\sqrt{3}$ is another root

The equation having roots

$2 \pm i\sqrt{3}$ *is* $x^2 - 4x + 7 = 0$

$\therefore x^2 - 4x + 7$ is a factor of

$x^4 - 4x^2 + 8x + 35$

	1	0	−4	8	35
4	–	4	16	20	–
−7	–	–	−7	−28	−35
	1	4	5	0	0

$x^2 + 4x + 5 = 0 \Rightarrow = \dfrac{-4 \pm \sqrt{16-20}}{2}$

$= \dfrac{-4 \pm 2i}{2} = -2 \pm i$

$\therefore$ The roots of the given equation are

$2 \pm i\sqrt{3}, -2 \pm i$

13. Solve the equation $x^4 - 6x^3 + 11x^2 - 10x + 2 = 0$, given that $2+\sqrt{3}$ is a root of the equation

Sol: $2+\sqrt{3}$ is one root $\Rightarrow 2-\sqrt{3}$ is another root. The equation having the roots of

$2 \pm \sqrt{3}$ *is* $x^2 - 4x + 1 = 0$

$\therefore x^2 - 4x + 1$ is factor of

$x^4 - 6x^3 + 11x^2 - 10x + 2 = 0$

	1	−6	11	−10	2
4	–	4	−8	8	–
−1	–	–	−1	2	−2
	1	−2	2	0	0

$x^2 - 2x + 2 = 0 \Rightarrow$

$x = \dfrac{2 \pm \sqrt{4-8}}{2} = \dfrac{2 \pm 2i}{2} = 1 \pm i$

$\therefore$ The roots of the required equation are $2 \pm \sqrt{3}, 1 \pm i$

14. Given that $-2+\sqrt{-7}$ is a root of the equation $x^4+2x^2-16x+77=0$, solve it completely

Sol: $-2-\sqrt{-7}\ (i.e)-2+i\sqrt{7}$ is one root $\Rightarrow -2+i\sqrt{7}$ is another root. The equation having the roots of $-2\pm i\sqrt{7}$ *is* $x^2+4x+11=0$

$\therefore x^2+4x+11$ is a factor of

$x^4+2x^2-16x+77$

	1	0	2	–16	77
– 4	–	– 4	16	– 28	–
– 11	–	–	–11	44	– 77
	1	– 4	7	0	0

$x^2-4x+7=0$

$x=\dfrac{4\pm\sqrt{16-28}}{2}=\dfrac{4\pm 2\sqrt{3}i}{2}$

$=2\pm\sqrt{3}i$

$\therefore$ The roots of the required equation are

$-2\pm i\sqrt{7},\ 2\pm\sqrt{3}i$

15 Solve the equation $x^4-9x^3+27x^2-29x+6=0$, given that one root of it $2-\sqrt{3}$

Sol: $2-\sqrt{3}$ is one root $\Rightarrow 2+\sqrt{3}$ is another root .

The equation having the roots of $2\pm\sqrt{3}$ is $x^2-4x+1=0$

$\therefore x^2-4x+1$ is a factor of the given equation

	1	– 9	27	– 29	6
4	–	4	– 20	24	0
–1	–	–	–1	5	– 6
	1	–5	6	0	0

$x^2 - 5x + 6 = 0 \Rightarrow (x-2)(x-3) = 0$

$x = 2, 3$

$\therefore$ The roots of the required equation are $2 \pm \sqrt{3}, 2, 3$

16. Show that the equation $\frac{a^2}{x-a'} + \frac{b^2}{x-b'} + \frac{x^2}{x-c'} + ... + \frac{k^2}{x-k'} = x - m$

Where a, b, c.....k, m, a' b', c'.....k' are all real numbers, cannot have a non-real root.

Sol: Let $\alpha + i\beta$ be a root of the given equation. Suppose if $\beta \neq 0$, then $\alpha - i\beta$ is also root of the given equation.

Substitute $\alpha + i\beta$ in the given equation, we get

$$\frac{a^2}{\alpha + i\beta - a'} + \frac{b^2}{\alpha + i\beta - b'} + ... + \frac{k^2}{\alpha + i\beta - k'}$$

$$= \alpha + i\beta - m$$

$$\Rightarrow \frac{a^2\left[(\alpha - a') - \beta\right]}{(\alpha - a')^2 + \beta^2} + \frac{b^2\left[(\alpha - b') - i\beta\right]}{(\alpha - b')^2 + \beta^2} + ... + \frac{k^2\left[(\alpha - k') - i\beta\right]}{(\alpha - k')^2 + \beta h2} = \alpha + i\beta - m \text{ ---------(1)}$$

Substitute $\alpha - i\beta$ in the given equation, we get

$$\frac{a^2}{\alpha - i\beta - a'} + \frac{b^2}{\alpha - i\beta - b'} + ... + \frac{k^2}{\alpha - i\beta - k'}$$

$$= \alpha - i\beta - m$$

$$\Rightarrow \frac{a^2\left[(\alpha - a') + i\beta\right]}{(\alpha - a')^2 + \beta^2} + \frac{b^2\left[(\alpha - b') + i\beta\right]}{(\alpha - b')^2 + \beta^2} + ... + \frac{k^2\left[(\alpha - k') + i\beta\right]}{(\alpha - k')^2 + \beta} = \alpha - i\beta - m \text{ --------------- (2)}$$

$(2) - (1) \Rightarrow 2i\beta$

$$\left[\frac{a^2}{(\alpha - a')^2 + \beta^2} + \frac{b^2}{(\alpha - b')^2 + \beta^2} + ... + \frac{k^2}{(\alpha - k')^2 + \beta^2}\right]$$

$= -2i\beta$

$\Rightarrow 2i\beta$

$$\left[\frac{a^2}{(\alpha - a')^2 + \beta^2} + \frac{b^2}{(\alpha - b')^2 + \beta^2} + \ldots\ldots \frac{k^2}{(\alpha - k')^2 + \beta^2} + 1\right]$$

$= 0$

$\Rightarrow \beta = 0$

This is a contradiction

$\therefore$ The given equation cannot have non-real roots.

17. Find the polynomial equation whose roots are the squares of the roots of $x^4 + x^3 + 2x^2 + x + 1 = 0$

Sol: Given equation is

$f(x) = x^4 + x^3 + 2x^2 + x + 1 = 0$

Required equation $f\left(\sqrt{x}\right) = 0$

$\Rightarrow x^2 + x\sqrt{x} + 2x + \sqrt{x} + 1 = 0$

$= \sqrt{x}(x+1) = -\left(x^2 + 2x + 1\right)$

Squaring both sides,

$\Rightarrow x(x+1)^2 = \left(x^2 + 2x + 1\right)^2$

$\Rightarrow x\left(x^2 + 2x + 1\right)$

$= x^4 + 4x^2 + 1 + 4x^3 + 4x + 2x^2$

$\Rightarrow x^3 + 2x^2 + x = x^4 + 4x^3 + 6x^2 + 4x + 1$

i.e., $x^4 + 3x^3 + 4x^2 + 3x + 1 = 0$

18. Form the polynomial equation whose roots are the squares of the roots of $x^3 + 3x^2 - 7x + 6 = 0$

Sol: Given equation is

$$f(x) = x^3 + 3x^2 - 7x + 6 = 0$$

Required equations is $f\left(\sqrt{x}\right) = 0$

$$\Rightarrow x\sqrt{x} + 3x - 7\sqrt{x} + 6 = 0$$

$$\Rightarrow \sqrt{x}\,(x-7) = -(3x+6)$$

Squaring on both sides

$$\Rightarrow x(x-7)^2 = (3x+6)^2$$

$$\Rightarrow x\left(x^2 - 14x + 49\right) = 9x^2 + 36 + 36x$$

$$\Rightarrow x^3 - 14x^2 + 49x - 9x^2 - 36x - 36 = 0$$

$$i.e., x^3 - 23x^2 + 13x - 36 = 0$$

19. Form the polynomial equation whose roots are cubes of the roots of $x^3 + 3x^2 + 2 = 0$

Sol: Given equation is $x^3 + 3x^2 + 2 = 0$

Put $y = x^2$ so that $x = y^{1/3}$

$$\therefore y + 3y^{2/3} + 2 = 0$$

$$3y^{2/3} = -(y+2)$$

Cubing on both sides, $27y^2 = -(y+2)^3$

$= -\left(y^3 + 6y^2 + 12y + 8\right)$

$\therefore y^3 + 6y^2 + 12y + 8 = 0$

$\Rightarrow y^3 + 33y^2 + 12y + 8 = 0$

Required equation is $x^3 + 33x^2 + 12x + 8 = 0$

20. **Solve** $x^3 - 3x^2 - 16x + 48 = 0$

Sol: Let $f(x) = x^3 - 3x^2 - 16x + 48$

By inspection, $f(3) = 0$

Hence 3 is a root of $f(x) = 0$

Now we divide $f(x)\, by\, (x-3)$

3	1	−3	−16	48
	–	3	0	−48
	1	0	−16	0

$\therefore x^2 - 16 = 0 \Rightarrow (x-4)(x+4) = 0$

$\Rightarrow x = -4, 4$

$\therefore$ The roots are $-4, 3, 4$

21. Find the roots of $x^4 - 16x^3 + 86x^2 - 176x + 105 = 0$

Sol: Let $f(x) = x^4 - 16x^3 + 86x^2 - 176x + 105$

Now, if $(1) = 1 - 16 + 86 - 176 + 105 = 0$

$\therefore$ 1 is a root of $f(x) = 0$

$\Rightarrow x - 1$ is a factor of $f(x)$

1	1	– 16	86	– 176	105
	–	1	– 15	71	– 105
	1	– 15	71	– 105	0

$\therefore f(x) = (x-1)(x^3 - 15x^2 + 71x - 105)$

$= (x-1)g(x)$ where

$g(x) = x^3 - 15x^2 + 71x - 105$

$g(1) = 1 - 15 + 71 - 105 = -48 \neq 0$

$g(2) = -15 \neq 0$

$g(3) = 27 - 135 + 213 - 105 = 0$

$\therefore 3$ is a root of $g(x) = 0$

$\Rightarrow x - 3$ is a factor of $g(x)$

3	1	– 15	71	– 105
	–	3	– 36	105
	1	– 12	35	0

$\therefore g(x) = (x-3)(x^2 - 12x + 35)$

$= (x-3)(x-5)(x-7)$

$\therefore f(x)=(x-1)(x-3)(x-5)(x-7)$

$\therefore 1, 3, 5, 7$ are the roots of $f(x)=0$

22. Solve $x^3-7x^2+36=0$, given one root being twice the other

Sol: Let α, β, γ be the root of the equation

$x^3-7x^2+36=0$ and

Let $\beta = 2\alpha$

Now we have $\alpha+\beta+\gamma=7$

$\Rightarrow 3\alpha+\gamma=7$ ------------- (1)

$\alpha\beta+\beta\gamma+\gamma\alpha=0$

$\Rightarrow 2\alpha^2+3\alpha\gamma=0$ ------------- (2)

$\alpha\beta\gamma=-36 \Rightarrow 2\alpha^2\gamma=-36$ ---------- (3)

From (1) and (2), we have

$2\alpha^2+3\alpha(7-3\alpha)=0$

i.e., $\alpha^2-3\alpha=0\,(or)\,\alpha(\alpha-3)=0$

$\therefore \alpha=0 \text{ or } \alpha=3$

Since $\alpha=0$ does not satisfy the given equation

$\therefore \alpha=3, \text{ so } \beta=6 \text{ and } \gamma=-2$

$\therefore$ The roots are 3, 6, -2

24. Given that 2 is a root of $x^3 - 6x^2 + 3x + 10 = 0$, find the other roots

Sol: Let $f(x) = x^3 - 6x^2 + 3x + 10$

Since 2 is a root of $f(x) = 0$, we divide $f(x)$ by $(x-2)$

	6	−13	−35	−1	3
4	–	24	44	12	–
−1	–	–	−6	−11	−3
	6	11	3	0	0

$\therefore x^3 - 6x^2 + 3x + 10 = (x-2)(x^2 - 4x - 5)$

$= (x-2)(x+1)(x-5)$

$\therefore -1, 2$ and 5 are the roots of the given equation

25. Give that two roots of $4x^3 + 20x^2 - 23x + 6 = 0$ are equal, find all the roots of the given equation

Sol: Let α, β, γ are the roots of $4x^3 + 20x^2 - 23x + 6 = 0$ -------1

Given two roots are equal, let α, β

W have $\alpha + \beta + \gamma = \frac{-20}{4} = -5$

$\Rightarrow 2\alpha + \gamma = -5$

$\Rightarrow \alpha\beta + \beta\gamma + \gamma\alpha = \frac{-23}{4}$

$\Rightarrow \alpha^2 + 2\alpha\gamma = \frac{-23}{4}$ ----------- (2)

$\alpha\beta\gamma = \frac{-6}{4} = \frac{-3}{2} \Rightarrow \alpha^2\gamma = \frac{-3}{2}$ ---------- (3)

From (1) and (2) $\alpha^2 + 2\alpha(-5-2\alpha) = \frac{-23}{4}$

$\Rightarrow 12\alpha^2 + 40\alpha - 23 = 0$

$\Rightarrow (2\alpha - 1)(6\alpha + 23) = 0$

$\alpha = \frac{1}{2}, \alpha = \frac{-23}{6}$

On verification, we get that

$\alpha = \frac{1}{2}$ is a root of (1)

$(2) \Rightarrow \gamma = -6$

$\therefore$ Roots are $\frac{1}{2}, \frac{1}{2}, -6$

26. Solve $4x^3 - 24x^2 + 23x + 18 = 0$**, given that the roots of this equation are in arithmetic Progression**

Sol: Let $a-d, a\ a+d$ are the roots of the given equation

Now, sum of the roots

$a - d + a + a + d = \frac{24}{4}$

$3a = 6$

$a = 2$

Product of the roots $(a-d)\,a(a+d) = \frac{-18}{4}$

$a(a^2 - d^2) = -\frac{9}{2}$

$2\left(4-d^2\right)=-\frac{9}{2}$

$4\left(4-d^2\right)=-9$

$16-4d^2=-9$

$4d^2=25$

$d=\pm\frac{5}{2}$

$\therefore$ Roots are $-\frac{1}{2}, 2$ *and* $\frac{9}{2}$

27. If the roots of $x^3+3px^2+3qx+r=0$, are in geometric progression, find the condition

Sol: The roots are in GP

Suppose the roots be $\frac{a}{R}, a, aR$

Given $\left(\frac{a}{R}\right)(a)(aR)=-R$

$\Rightarrow a^3=-1$

$\Rightarrow a=(-r)^{1/3}$

$\because$ 'a' is a root of $x^3+3px^2+3qx+r=0$

$\Rightarrow\left(-r^{1/3}\right)^3+3p\left(-r^{1/3}\right)^2+3q\left(-r^{1/3}\right)+r=0$

$\Rightarrow -r+3pr^{2/3}-3qr^{1/3}+r=0$

$pr^{2/3}=qr^{1/3}$

$\Rightarrow pr^{1/3}=q$

$\Rightarrow p^3r=q$ is the required condition